JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 111, C08004, doi:10.1029/2005JC003407, 2006

Return period of nonlinear high wave crests

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Received 21 November 2005; revised 1 May 2006; accepted 18 May 2006; published 1 August 2006.

[1] This paper deals with the long-term statistics for extreme nonlinear crest heights. First, a new analytical solution for the return period R(η), of a sea storm in which the maximum nonlinear crest height exceeds a fixed threshold η, is obtained by applying the 'Equivalent Triangular Storm' model and a second-order crest height distribution. The probability $P(\eta_{c\ \max} > \eta|[0, L])$ that maximum nonlinear crest height in the time span L exceeds a fixed threshold is then derived from R(η) solution, assuming that the occurrence of storms with highest crest larger than η is given by a Poisson process. In the applications, both R(η) and $P(\eta_{c\ \max} > \eta|[0, L])$ are calculated for some locations. It is shown that narrowband second-order approach is slightly conservative, with respect to the more general condition of crest distribution for second-order three-dimensional waves. Finally, a comparison with Boccotti, Jasper and Krogstad models is presented.

Citation: Arena, F., and D. Pavone (2006), Return period of nonlinear high wave crests, *J. Geophys. Res.*, *111*, C08004, doi:10.1029/2005JC003407.

1. Introduction

[2] The height of extreme waves during storms is usually obtained by investigating both the statistical properties of waves in a sea state (short-term statistics) and the distribution of the significant wave height at the specified location (long-term statistics).

[3] As for the short-term statistics, to the first-order in a Stokes expansion, the surface displacement is a random Gaussian process of time, and crest-to-trough heights have the Rayleigh distribution for an infinitely narrow spectrum [*Longuet-Higgins*, 1952]. For finite bandwidth the largest crest-to-trough heights tend to a Weibull distribution [*Boccotti*, 1981, 1997, 2000], which depends upon the bandwidth of the frequency spectrum [*Longuet-Higgins*, 1980; *Boccotti*, 1982; *Forristall*, 1984].

[4] Nonlinearities modify both the crest and the trough distributions in a sea state: the Rayleigh law, which is exact to the first-order in a Stokes expansion for a narrowband spectrum and gives also the asymptotic crest height distribution for a Gaussian sea state with finite bandwidth of the spectrum [see *Boccotti*, 2000], tends to underestimate the crest height and to overestimate the trough amplitude of the surface displacement.

[5] The effects of nonlinearity were first investigated by *Longuet-Higgins* [1963]. A second-order model for the probability distribution of the crest height for the narrowband surface displacement was proposed by *Tayfun* [1980] and *Tung and Huang* [1985]. A more general narrowband second-order approach was proposed by *Arena and Fedele* [2002a], that obtained the crest and the trough distributions for a stochastic family, which includes many processes in the mechanics of the sea waves.

[6] The second-order crest height distribution, for the more general condition of three-dimensional waves (that is including effects of both finite bandwidth and directional spreading function) was finally given by *Forristall* [2000]. This distribution gives results in good agreement with both field data [*Prevosto and Forristall*, 2004] and other second-order crest height models [*Prevosto et al.*, 2000; *Al-Humoud et al.*, 2002; *Fedele and Arena*, 2005].

[7] As for the long-term statistics, *Isaacson and Mackenzie* [1981], *Guedes Soares* [1989] and *Goda* [1999] gave some complete reviews. In many locations, three-parameter Weibull law fits well extreme significant wave height values [see, e.g., *Battjes*, 1970; *Burrows and Salih*, 1986; *Ochi*, 1998; *Boccotti*, 2000; *Arena and Barbaro*, 1999; *Arena*, 2004].

[8] In this paper a new analytical solution for the return period $R(\eta)$ of a sea storm in which the maximum nonlinear crest height exceeds a fixed threshold η is obtained. The solution is based on the 'Equivalent Triangular Storm' model [*Boccotti*, 1986, 2000] that associates two parameters to each actual storm: the triangle height (storm intensity) and the triangle base (storm duration). It is shown that $R(\eta)$ depends on the crest height distribution, the long-term statistics and the mean value of the duration of the equivalent triangular storms.

[9] Therefore, for some locations in the Central Mediterranean sea and in the Pacific Ocean, $R(\eta)$ as well as the probability that maximum crest height during a large time span exceeds a fixed threshold, is calculated and both the effects of nonlinearity and of finite bandwidth are analyzed. A comparison with *Boccotti* [1986, 2000], *Jasper* [1956]

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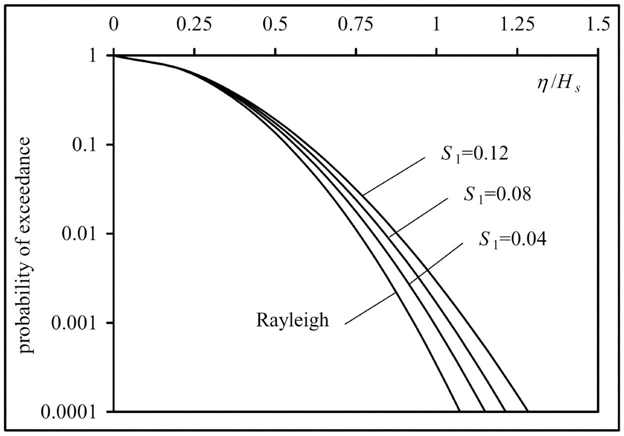

Figure 1. Second-order probability of exceedance of the crest height (equation (1)) in deep water ($d \to \infty$) for fixed values of the wave steepness S_1 [the Rayleigh distribution is given by equation (1), as $S_1 \to 0$].

and *Krogstad* [1985; see also *Krogstad and Barstow*, 2004] models is finally presented.

2. Second-Order Distribution of the Crest Heights in a Sea State

[10] To first order the free surface displacement is a linear Gaussian process and, if we assume that the spectrum is narrowband, the crests and the troughs are Rayleigh distributed. For finite bandwidth of the spectrum, the Rayleigh law gives the asymptotic linear crest height distribution also. Actually, because of nonlinearity, the Rayleigh law tends to under-predict the crest amplitude and to over-predict the trough amplitude.

[11] In this paper *Forristall*'s [2000] perturbated Weibull model, for the second-order crest height distribution, is considered. He proposed a two parameters Weibull law for the probability of exceedance of the crest height, which is defined as the probability that a crest height is greater than η in a sea state with significant wave height H_s:

$$P(\eta_c > \eta) = \exp\left[-\left(\frac{\eta}{\alpha H_s}\right)^{\beta}\right]. \tag{1}$$

For three-dimensional waves, the parameters α and β are given respectively by:

$$\alpha = 1/\sqrt{8} + 0.2568 S_1 + 0.0800 U_r \tag{2}$$

$$\beta = 2 - 1.7912 S_1 - 0.5302 U_r + 0.2824 U_r^2 \tag{3}$$

where

$$S_1 = \frac{2\pi}{g}\frac{H_s}{T_{m01}^2} \tag{4}$$

is a wave steepness parameter, and

$$U_r = \frac{H_s}{k_1^2 d^3} \tag{5}$$

is the Ursell number on the water depth d.

[12] In the previous equations,

$$T_{m01} = 2\pi \frac{m_0}{m_1} \tag{6}$$

is the mean wave period,

$$k_1 = \frac{(2\pi)^2}{g T_{m01}^2} \tag{7}$$

is the wave number in deep water corresponding to T_{m01} and

$$m_j = \int_0^\infty \omega^j E(\omega)\mathrm{d}\omega \tag{8}$$

defines the jth moment of the spectrum.

[13] Figure 1 shows the probability of exceedance (1) in deep water ($d \to \infty$, which implies $U_r \to 0$), for fixed values of S_1; let us note that distribution (1) tends to the Rayleigh law as the wave steepness S_1 tends to zero.

[14] As regards the wave steepness S_1 and the Ursell number U_r for wind-waves spectra, the jth moment for a JONSWAP spectrum [*Hasselmann et al.*, 1973] is rewritten as $m_j = \alpha g^2 \omega_p^{-4+j} m_{w_j}$, where ω_p is the peak frequency, α is the Phillips' parameter and m_{w_j} is the nondimensional jth moment, which is given by:

$$m_{w_j} = \int_0^\infty w^{-5+j}\exp\left(-1.25 w^{-4}\right)\exp\left\{\ln\gamma \exp\left[-\frac{(w-1)^2}{2\sigma^2}\right]\right\}\mathrm{d}w \tag{9}$$

where σ may be assumed equal to 0.08 and γ is equal to 3.3 for the mean JONSWAP spectrum and to 1.0 for the Pierson-Moskowitz spectrum [*Pierson and Moskowitz*, 1964].

[15] The mean period T_{m01} is then

$$T_{m01} = T_p m_{w_0}/m_{w_1} \tag{10}$$

and the ratio m_{w_0}/m_{w_1} is equal to 0.84 for the mean JONSWAP and to 0.78 for the Pierson-Moskowitz spectrum.

[16] Furthermore, because peak period T_p, for the JONSWAP spectrum, may be calculated as a function of H_s:

$$T_p = \pi (H_s/g)^{0.5}/(m_{w0}\alpha)^{0.25}, \tag{11}$$

the wave steepness S_1 (equation (4)) may be rewritten as:

$$S_1 = \frac{2\alpha^{0.5} m_{w_1}^2}{\pi m_{w_0}^{1.5}}. \tag{12}$$

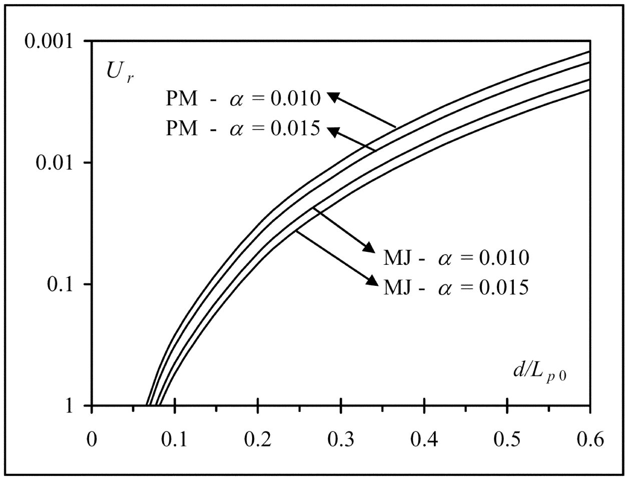

Figure 2. Ursell number U_r versus d/L_{p_0}, for the mean JONSWAP spectrum (MJ) and the Pierson-Moskowitz spectrum (PM), calculated for α equal to 0.010 and to 0.015.

[17] For the mean JONSWAP spectrum, S_1 ranges from 0.044 to 0.070 for α ranging from 0.008 to 0.02 respectively. For α = 0.01, S_1 is equal to 0.050 for the mean JONSWAP spectrum and to 0.047 for the Pierson-Moskowitz spectrum.

[18] The Ursell number, given by equation (5), for the JONSWAP spectrum may be rewritten as

$$U_r = \frac{\alpha^{0.5} m_{w_0}^{4.5}}{2\pi^3 m_{w_1}^4} \frac{1}{\left(d/L_{p_0}\right)^3} \tag{13}$$

where $L_{p_0} \equiv gT_p^2/(2\pi)$. Figure 2 shows the Ursell number vs. d/L_{p_0}, for both the mean JONSWAP and the Pierson-Moskowitz spectrum, for fixed values of Phillips' parameter α.

3. Equivalent Triangular Storm (ETS) Model

[19] A sea storm may be defined as a sequence of sea states in which the significant wave height exceeds the threshold h_{crit} and does not fall below this value for a continuous time interval longer than 12 hours [*Boccotti*, 2000]; as for h_{crit}, the value of 1.5 times $\overline{H_s}$ was proposed [see also *Arena and Barbaro*, 1999], where $\overline{H_s}$ is the mean annual significant wave height at the examined location.

[20] The probability of exceedance of the crest height varies with the sea state during the storm. If we assume that crest heights are stochastically independent each other, the probability of exceedance of the maximum wave crest during a sea storm whose duration is D may be written:

$$P(\eta_{c\max} > \eta|[0, D]) = 1 - \exp\left\{\sum_{i=1}^{N} \frac{\Delta t_i}{T_{z_i}} \ln[1 - P_i(\eta_c > \eta)]\right\} \tag{14}$$

where, for the ith sea state of the storm, P_i $(\eta_c > \eta)$ is the crest exceeding probability, Δt_i, T_{z_i} and Δ_i/T_{z_i} are respectively the duration, the mean zero-crossing period and the number of zero-crossing waves. If the actual period T_{z_i} is unknown, it may be replaced by the average value calculated as a function of the significant wave height (see section 5.2). Note that N defines the number of sea states in the storm, so that $D \equiv \sum_{i=1}^{N} \Delta t_i$. The limit of (14) as Δt_i tends to zero, is [*Borgman*, 1970, 1973]:

$$P(\eta_{c\max} > \eta|[0, D]) = 1 - \exp\left\{\int_0^D \frac{1}{T_z} \ln[1 - P(\eta_c > \eta)]\mathrm{d}t\right\}, \tag{15}$$

where both $P(\eta_c > \eta)$ and T_z are time dependent.

[21] The ETS model [*Boccotti*, 1986, 2000] associates two parameters to each actual sea storm: the height and the base of a triangle (see Figure 3). The triangle height a, which represents the storm intensity, is equal to the maximum significant wave height during the actual storm; the triangle base b, which represents the storm duration, is such that the maximum expected wave height of the triangular

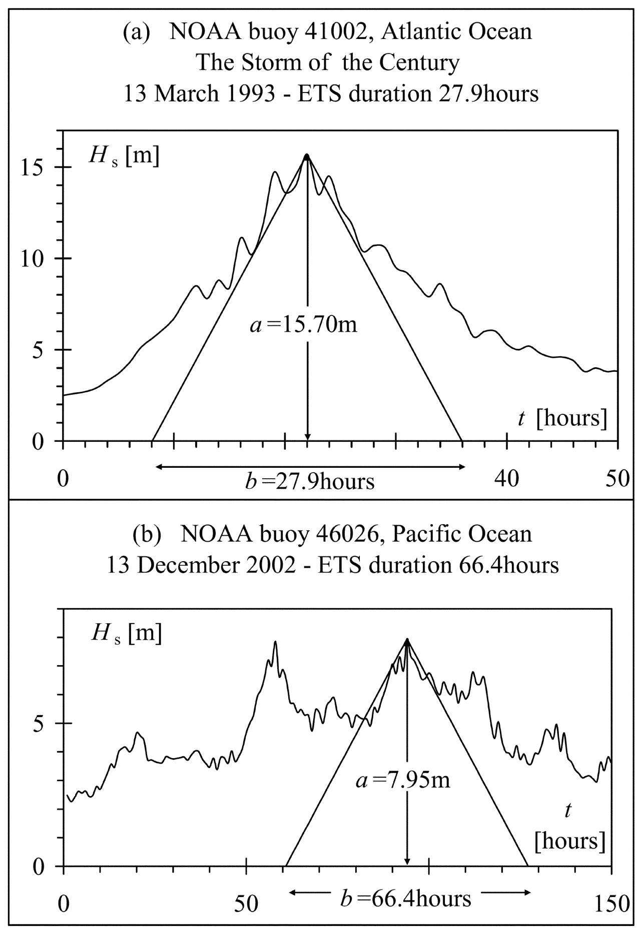

Figure 3. Actual storms with associated equivalent triangular storms. (a) Storm of the century, recorded by buoy NOAA 41002 in Atlantic Ocean; (b) strongest sea storm recorded by buoy 46026.

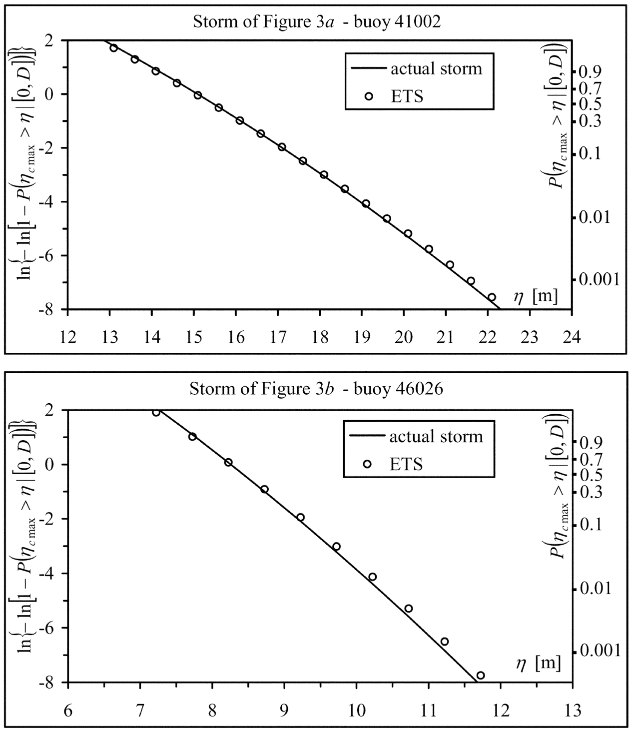

Figure 4. Comparison between probability of exceedance $P(\eta_{c\ \max} > \eta|[0, D])$ of actual storm (continuous lines) and probability of exceedance $P(\eta_{c\ \max} > \eta|[0, b])$ of corresponding ETS (circles), for the two storms of Figure 3.

storm is equal to the maximum expected wave height of the actual storm. The probability of exceedance (15) for a triangular storm with height a and base b becomes:

$$P(\eta_{c\max} > \eta|[0,b]) = 1 - \exp\left\{\frac{b}{a}\int_0^a \frac{1}{T_z}\ln[1 - P(\eta_c > \eta)]\mathrm{d}h\right\}. \tag{16}$$

where h denotes the significant wave height, that linearly varies during the ETS, and T_z is the mean period which is evaluated as a function of h, either by means of a theoretical relation or by processing the data of the joint occurrence of significant wave heights and mean wave periods at the given location (see section 5.2).

[22] *Boccotti* [2000], *Arena and Barbaro* [1999] and *Arena and Fedele* [2002b] by processing data of hundreds of actual sea storms, showed that statistical properties of each sea storm and its associated ETS are very close each other.

[23] For example Figure 4 shows, for the two storms of Figure 3, a good agreement between probabilities $P(\eta_{c\ \max} > \eta|[0, D])$ of actual storm (continuous lines) and $P(\eta_{c\ \max} > \eta|[0, b])$ of corresponding ETS (circles). Both the above extreme crest distributions are nearly straight lines in the plot scale of Figure 4, therefore they are very close to being Gumbel distributions [see *Borgman*, 1973; *Krogstad and Barstow*, 2004]. Moreover, because the parameters of the Gumbel distribution depend only on the mean and the standard deviation of $\eta_{c\ \max}$, they are very close each other if calculated from the actual storm or the associated ETS.

4. Long-Term Distribution of Significant Wave Height

[24] The long-term distribution of extreme significant wave heights, in many locations, is well fitted by three-parameter (lower bounded) Weibull law. This law although is able to model well the tail of the distribution, often fails to represent the whole distribution [*Ferreira and Guedes Soares*, 2000]. An approach adopted in many cases is to use only the low probability part of the data to fit the Weibull distribution [*Guedes Soares*, 1986; *Boccotti*, 2000] or even to combine a Weibull for the low probability region with a log-normal distribution for the high probability one, as proposed by *Haver* [1985].

[25] In this paper the buoy data of three locations are processed for the applications. The first location is Ponza island, in Central Mediterranean Sea, where a directional pitch-roll buoy is moored in deep water. This buoy, which belongs to the RON Italian buoys network, of the Agency for Environmental Protection and Technical Services (APAT), gives values of the significant wave height, mean period, peak period and wave direction every third hour (buoy data and information are available at http://www.idromare.com). The processed data cover the time span July 1989 – June 2003.

[26] The other two locations are off the west coast of the United States: the NOMAD buoy 46002 is moored off Oregon, at water depth of 3374 m; the discus buoy 46026 is moored off San Francisco (California), at water depth of 52.1 m. These buoys, which belong to the NOAA- National Data Buoy Center (United States), give values of significant wave height, mean period, peak period and wave direction every hour (buoy data and information are available at http://www.ndbc.noaa.gov). The processed data cover the years 1975–2003 for the buoy 46002 and 1982–2003 for the buoy 46026.

[27] For the examined buoys, the probability of exceedance of the extreme significant wave heights $P(H_s > h)$ is well fitted by the three-parameter Weibull law:

$$P(H_s > h) = \exp\left[-\left(\frac{h - h_l}{w}\right)^u\right] \tag{17}$$

where the shape parameter u, the location parameter h_l and the scale parameter w are given in Table 1.

5. Return Period $R(\eta)$ of a Sea Storm in Which the Maximum Nonlinear Crest Height Exceeds the Fixed Threshold η

[28] The concept of return period of sea storms exceeding a fixed threshold was first discussed by *Borgman* [1963]. In this paper a new solution is obtained for the return period $R(\eta)$, which is defined as the time period that has, on the average, one storm in which the maximum crest height

Table 1. Mean Significant Wave Height $\overline{H_s}$, the Mean ETS Base $\bar{b}$ and the Weibull $P(H_s > h)$ Parameters h_l, w, u (Equation (17)) for Some Locations

Buoy Location	$\bar{b}$, hour	$\overline{H_s}$, m	u	w, m	h_l, m
RON Ponza 40.9°N–13.0°E	77	0.85	1.057	0.740	0.08
NOAA 46002 42.5°N–130.3°W	65	2.69	1.253	1.784	1.02
NOAA 46026 37.8°N–122.8°W	58	1.79	1.378	1.249	0.65

exceeds the fixed threshold η. If we consider a very large time span T, $R(\eta)$ is given by:

$$R(\eta) = \frac{\mathrm{T}}{\mathrm{N}(\eta_{c\max} > \eta|[0, \mathrm{T}])}, \tag{18}$$

where $\mathrm{N}(\eta_{c\ \max} > \eta|[0, \mathrm{T}])$ gives the number of sea storms, during the time span T, in which the maximum crest height exceeds the threshold η.

[29] To obtain the return period $R(\eta)$ the ETS model is applied by substituting a triangle to each of the N(T) actual storms which occur during the time span T. If we denote with $p_A(a)$ the probability density function of the triangle heights and $p_B(b)$ the probability density function of the triangle bases (we assume that the triangle bases and heights are stochastically independent each other), we have that

$$p_A(a)\, p_B(b)\, \mathrm{d}a\, \mathrm{d}b \mathrm{N}(\mathrm{T}) \tag{19}$$

defines the number of ETSs with height within (a, a + da) and base within (b, b + db), during T. The number of ETSs during T, with height within (a, a + da) and base within (b, b + db), in which the maximum crest height exceeds the fixed threshold η is then:

$$p_A(a)\, p_B(b) \mathrm{d}a\, \mathrm{d}b \mathrm{N}(\mathrm{T}) P(\eta_{c\max} > \eta|[0, b]). \tag{20}$$

[30] From equation (20), by integrating, we achieve

$$\mathrm{N}(\eta_{c\max} > \eta|[0, \mathrm{T}]) = \mathrm{N}(\mathrm{T}) \int_0^\infty \int_0^\infty p_A(a)\, p_B(b) \cdot P(\eta_{c\max} > \eta|[0, b]) \mathrm{d}b\, \mathrm{d}a \tag{21}$$

Note that the number of sea storms during T in which the maximum crest height exceeds the threshold η is a stochastic variable that converges to the value $\mathrm{N}(\eta_{c\ \max} > \eta|[0, \mathrm{T}])$, given by equation (21), as T increases.

[31] In order to obtain the analytical expression of the probability density function $p_A(a)$, we consider the time $\Delta t(h, \mathrm{d}h, \mathrm{T})$ during T, in which the significant wave height H_s is within a small fixed interval (h, h + dh). It is given by:

$$\Delta t(h, \mathrm{d}h, \mathrm{T}) = \mathrm{T} p_{H_S}(h) \mathrm{d}h \tag{22}$$

If we consider the equivalent sea, the time $\Delta t(h, \mathrm{d}h, \mathrm{T})$ may be written as:

$$\Delta t(h, \mathrm{d}h, \mathrm{T}) = N(\mathrm{T}) \int_0^\infty \int_0^\infty p_A(a) p_B(b) \delta t(h, \mathrm{d}h, a, b) \mathrm{d}b \mathrm{d}a \tag{23}$$

where $\delta t(h, \mathrm{d}h, a, b)$ denotes the time in which the significant wave height is within the interval (h, h + dh) during an equivalent triangular storm whose maximum significant height is between a and a + da and whose duration is between b and b + db; this time is zero if $h \geq a$ and is equal to $b/a\mathrm{d}h$ elsewhere.

[32] From equations (22)–(23) after some algebra we get [*Boccotti*, 2000]:

$$p_A(a) = -\frac{\mathrm{T}}{\mathrm{N}(\mathrm{T})} \frac{a}{\bar{b}} \frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a}. \tag{24}$$

From equations (18), (21) and (24) it follows that

$$R(\eta) = -\left\{ \int_0^\infty \frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a} \frac{a}{\bar{b}} \int_0^\infty p_B(b) \cdot \left[1 - \exp\left(\frac{b}{a} \int_0^a \frac{1}{T_z(h')} \ln[1 - P(\eta_c > \eta)]\, \mathrm{d}h' \right) \right] \mathrm{d}b \mathrm{d}a \right\}^{-1}. \tag{25}$$

[33] Here we will assume the $p_B(b)$ distribution given by the delta function, which is slightly conservative [*Boccotti*, 2000]: in words, we consider that all the bases b of the triangles are equal to the mean value $\bar{b}$. This assumption will be justified in section 5.3 where the return period is calculated with the probability density function $p_B(b)$ of actual bases, obtained by processing buoys data. Finally we have:

$$R(\eta) = -\left\{ \int_0^\infty \frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a} \frac{a}{\bar{b}} \cdot \left[1 - \exp\left(\frac{\bar{b}}{a} \int_0^a \frac{1}{T_z(h')} \ln[1 - P(\eta_c > \eta)]\, \mathrm{d}h' \right) \right] \mathrm{d}a \right\}^{-1}. \tag{26}$$

Therefore, the return period $R(\eta)$ depends on the short-term statistics, and in particular the mean zero crossing period T_z (see section 5.2) and the crest height distribution $P(\eta_c > \eta)$. It also depends on the mean base $\bar{b}$ of the triangles and the probability density function of the significant wave height $p_{H_s}(a)$.

[34] Let us note that $p_{H_s}(a)$ is positive only if

$$\frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a} < 0. \tag{27}$$

Assuming the probability of exceedance of the significant wave height given by the lower-bounded three-parameter Weibull (which is defined for $H_s > h_l$, see equation (17)), we have that inequality (27) is satisfied for $h > h^*$, where

$$h^* = h_l + w\left(1 - \frac{1}{u}\right)^{\frac{1}{u}}. \tag{28}$$

In many locations it was found, by numerical investigation [*Arena*, 2004], that h^* is always smaller than the sea storms

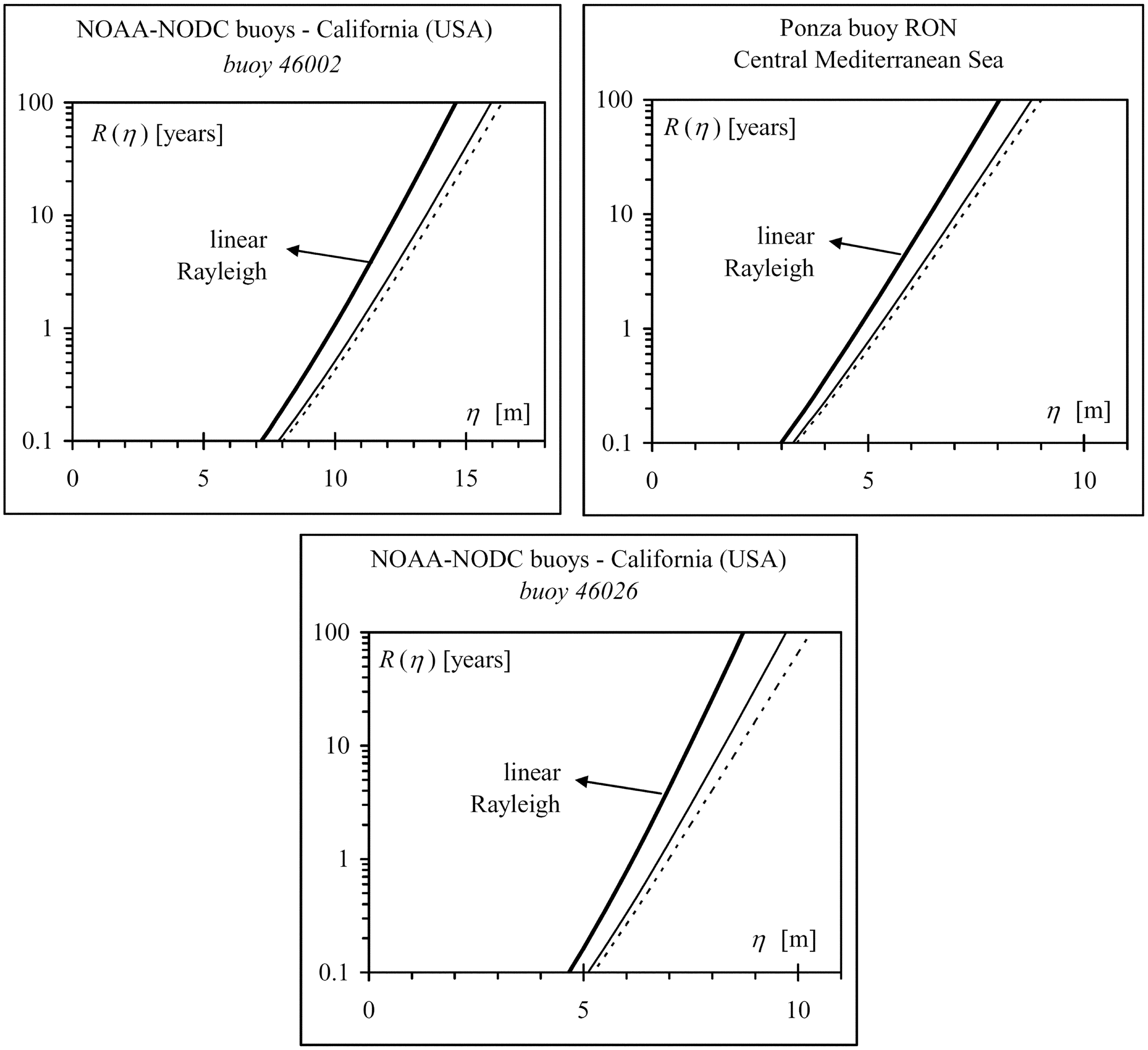

Figure 5. Return period $R(\eta)$ of a sea storm in which highest crest height exceeds threshold η: comparison between linear and second-order models. Continuous lines show nonlinear prediction for three-dimensional waves (that is, for the crest height distribution given by equation (1)). Dotted lines show nonlinear prediction for narrowband crest height distribution.

threshold h_{crit}. For the three examined buoys it is easy to verify that $h^* < h_{crit}$: from Table 1 we find that h^* is equal to 0.13 m, 1.52 m and 1.14 m for PONZA, 46002 and 46026 buoys respectively. We may conclude that

$$\frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a} < 0 \text{ for } a > h_{crit} \tag{29}$$

and therefore inequality (27) gives no restriction on the calculation of $R(\eta)$ (note that the lower limit in the first integral of equation (26) may be assumed equal to h_{crit}).

5.1. Calculation of $R(\eta)$ for Nonlinear Crest Height Distribution

[35] In the calculation of the return period $R(\eta)$ we adopt the Forristall second-order crest height distribution, given by equation (1). Therefore the probability of exceedance of the crest height $P(\eta_c > \eta)$ is given as a function of the wave steepness S_1 and the Ursell number U_r.

[36] In deep water, the Ursell number tends to zero and the wave steepness, for a fixed value of the Phillips' parameter and a given spectrum, may be assumed constant (see equation (12)). In this condition, both the parameters α and β of crest height distribution (equation (1)) are constant whatever H_s value is. As a consequence, in deep water, the return period $R(\eta)$ may be calculated with equation (26), assuming that α and β parameters do not depend upon variables of integration a and h'.

[37] On a finite depth d, we have that both S_1 and U_r depend upon significant wave height (note that the variability of S_1 may be given, for example, by shoaling-refraction effects). Therefore, by fixing the d value (which could be, for example, the water depth in which a buoy is moored), in

Table 2. Crest Amplitudes η for Fixed Threshold of the Return Period R: Comparison Among η (R) Values Calculated for Crest Height Distribution Given by Linear Model, Narrowband Second-Order Model, and Second-Order Three-Dimensional Wave Model (Equation (1))

	Buoy	Linear $\eta(R)$, m	Second-Order Narrowband $\eta(R)$, m	Second-Order 3-D Waves $\eta(R)$, m
R = 1 year	RON -Ponza	4.8	5.3	5.2
	NOAA-46026	6.2	7.0	6.8
	NOAA-46002	9.8	11.0	10.7
R = 100 years	RON -Ponza	8.0	9.0	8.7
	NOAA-46026	8.7	10.3	9.7
	NOAA-46002	14.5	16.3	15.8

order to calculate $R(\eta)$ we have to consider that both S_1 and U_r (and therefore α and β) depend on the variables of integration a and h' (see equation (26)). As for the mean zero-crossing period T_z, it is investigated in next section.

[38] Figure 5 shows the return period $R(\eta)$ in three locations in Central Mediterranean Sea and in Pacific Ocean (see section 4), calculated for the Rayleigh crest height distribution, to the second-order for narrowband spectrum (dotted lines), and for the more general condition of three-dimensional waves (continuous lines). Furthermore, Table 2 shows the crest amplitudes for fixed thresholds of the return period (equal to 1 and 100 years).

[39] Following linear theory, the heights of extreme crests are smaller than those given by three-dimensional second-order crest distribution: for a fixed value of the return period the differences are close to 8% in deep water (Ponza and 46002 buoys), and they are between 9 and 11% for the buoy 46026 (moored at depth d = 52.1 m). Note that for the buoy at finite depth the differences between first and second-order models tend to increase as the value R of the return period increases.

[40] As for effects of finite bandwidth and of directional spreading, the comparison between second-order models for three-dimensional waves and for narrowband spectrum shows that the latter approach is slightly conservative.

[41] Finally, it should be noted that the applications concern sea states with single-peaked spectra. This should be not a limitation for calculation of extreme crest heights, because with large probability they will occur in high sea states, when double-peaked spectra do not happen so often [*Guedes Soares*, 1984]. Anyway the presented results may be applied also for combined sea states, which have spectra that exhibit more than one peak [see *Guedes Soares*, 1991]. For this purpose the distribution of nonlinear crest heights should be obtained for double-peaked spectra (at present statistical properties of individual waves in combined sea states have been investigated only to the first order [*Rodriguez et al.*, 2002]).

5.2. Mean Zero Crossing Period T_z

[42] The theoretical mean zero crossing period [*Rice*, 1944, 1945], which is defined as

$$T_z = 2\pi\sqrt{m_0/m_2}, \tag{30}$$

may be rewritten as a function of nondimensional moments defined in section 2:

$$T_z = T_p\sqrt{m_{w_0}/m_{w_2}} \tag{31}$$

(where $\sqrt{m_{w_0}/m_{w_2}}$ is equal to 0.79 for a mean JONSWAP and to 0.73 for a Pierson-Moskowitz spectrum) or, as a function of significant wave height:

$$T_z = m_{w_0}^{0.25} m_{w_2}^{-0.5}\alpha^{-0.25}\pi\left(H_s/g\right)^{0.5}. \tag{32}$$

For a mean JONSWAP spectrum and for α = 0.01, equation (32) may be rewritten as T_z = 10.6$(H_s/g)^{0.5}$. Actual data show that this theoretical T_z value occurs for strongest sea states only [*Arena*, 2004]. In general, measured mean wave period T_z is greater than theoretical value (32). For example, for NOAA buoy 46002 and for RON buoy Ponza (see section 4) the regression of actual mean periods (derived for sea states with $H_s > h_{crit}$) is given by:

$$T_z = \chi_T 10.6\left(H_s/g\right)^{0.5}, \text{ with } \chi_T = C_1\ln\left(H_s/\overline{H_s}\right) + C_2 \tag{33}$$

where the mean significant wave height $\overline{H_s}$ values are shown in Table 1 and (C_1, C_2) coefficients are equal to (−0.181,1.258) and to (−0.150,1.331) for the 46002 and Ponza buoys respectively.

[43] However, it is found, by numerical investigation, that the assumption of the theoretical value of mean zero-crossing period T_z, in place of actual regression given by equation (33), has negligible consequence on calculation of extreme crests.

5.3. Effects of Distribution of ETS Bases for the Calculation of $R(\eta)$

[44] To validate the assumption of delta function for the base distribution, the data of the buoys shown in section 4 are processed to obtain the probability density function $p_B(b)$ of actual bases. The return period for the NOAA 46002 and RON Ponza buoys is then calculated by means of equation (25), with actual $p_B(b)$ given in Figure 6. Results are compared with those given by equation (26). The differences for the crest heights, for given values of the return period R, are smaller than 0.3%. For example, the crest height in the buoy 46002 for return period of 100

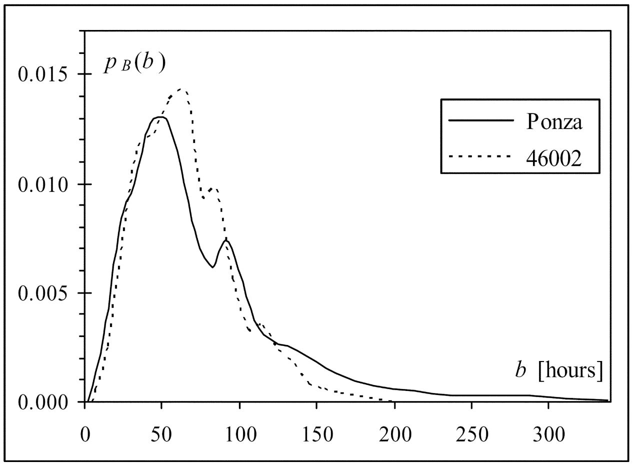

Figure 6. Probability density function of bases of equivalent triangular storms for the RON Ponza and the NOAA-NODC 46002 buoys.

years, is equal to 15.84 m if $p_B(b) = \delta(b - \bar{b})$ (see Table 2) and to 15.81 m for the actual probability density function $p_B(b)$ of Figure 6.

[45] Finally, as hypothesis test of the stochastic independence of the ETS heights and bases, the correlation coefficient is calculated, which is defined as:

$$\rho_{a,b} = \frac{\sum_{i=1}^{N} (a_i - \bar{a})(b_i - \bar{b})}{N\sigma_a\sigma_b}, \tag{34}$$

where ($\bar{a}$, σ_a) and ($\bar{b}$, σ_b) are respectively the mean value and the standard deviation of the N triangle heights a and bases b. The coefficient $\rho_{a,b}$ is equal to −0.098, −0.003, −0.060 for the buoys 46002, 46022, Ponza respectively.

6. Probability That Maximum Nonlinear Crest Height, in the Lifetime L of a Structure, Exceeds the Fixed Threshold η

[46] The probability $P(\eta_{c\ \max} > \eta|[0, L])$ that the maximum crest height, during the time span L, exceeds a fixed threshold η is equal to the probability that at least one storm, in which the largest crest height exceeds η, occurs during L. Therefore assuming that the occurrences of storms with the highest crest larger than η form a homogeneous Poisson process, we have:

$$P(\eta_{c\max} > \eta|[0, L]) = 1 - \exp\left[-\frac{L}{R(\eta)}\right] \tag{35}$$

and, from expression (26)

$$P(\eta_{c\max} > \eta|[0, L]) = 1 - \exp\left\{L\int_0^\infty \frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a}\frac{a}{\bar{b}}\cdot \left[1 - \exp\left(\frac{\bar{b}}{a}\int_0^a \frac{1}{T_z(h')} \cdot \ln\left[1 - P(\eta_c > \eta)\right]\mathrm{d}h'\right)\right]\mathrm{d}a\right\}. \tag{36}$$

[47] Figure 7 shows probability (36) for NOAA buoy 46002, calculated for L equal to 10 and 100 years.

7. Comparison With Other Models

7.1. Boccotti's Solution for the Return Period $R(H)$

[48] The return period $R(H)$ of a storm whose maximum wave height exceeds the fixed threshold H was obtained by *Boccotti* [1986, 2000], for the 'equivalent sea'. Following his logic, the return period $R(\eta)$ may be written as

$$R(\eta) = \left\{-\int_\eta^\infty\int_0^\infty \frac{p_{\eta_c}(\eta')}{1 - P(\eta_c > \eta')}\frac{1}{T_z(h)}\int_h^\infty \frac{\mathrm{d}p_{H_s}(a)}{\mathrm{d}a} \cdot \exp\left[\frac{\bar{b}(a)}{a}\int_0^a \frac{\ln[1 - P(\eta_c > \eta')]}{T_z(h')}\mathrm{d}h'\right]\mathrm{d}a\mathrm{d}h\mathrm{d}\eta'\right\}^{-1}. \tag{37}$$

[49] Numerical investigations show that, for any fixed value of R, there are not significant differences between $\eta(R)$ values given by equation (37) and by the simpler new expression (26).

7.2. Jasper's Solution for the Return Period $R_J(H)$ of a Wave Height Exceeding a Fixed Threshold H

[50] The solution for the return period $R_J(H)$ of a 'wave height exceeding a fixed threshold H' was given by *Jasper* [1956]. His solution, which has an easy formal derivation, may be particularized for the 'crest height exceeding a fixed threshold η', giving

$$R_J(\eta) = \left[\int_0^\infty P(\eta_c > \eta)\frac{p_{H_s}(h)}{T_z(h)}\mathrm{d}h\right]^{-1}. \tag{38}$$

7.3. Krogstad's Expression for the Probability $P(\eta_{c\ \max} > \eta|[0, L])$ That the Maximum Crest Height, During the Time Span L, Exceeds the Threshold η

[51] *Krogstad* [1985] proposed a different approach, based on the *Borgman* [1963] integral relation, to achieve the probability $P(\eta_{c\ \max} > \eta|[0, L])$ that the maximum crest height, during the time span L, exceeds the threshold η:

$$P(\eta_{c\max} > \eta|[0, L]) = 1 - \exp\left\{L\int_0^\infty p_{H_s}(h)\frac{1}{T_z(h)}\ln[1 - P(\eta_c > \eta)]\mathrm{d}h\right\}. \tag{39}$$

Figure 7 shows, for the NOAA buoy 46002, the comparison between probabilities $P(\eta_{c\ \max} > \eta|[0, L])$ obtained with equations (36)–(39) (presented model, continuous line, and Krogstad model, dotted line, respectively). We can see that Krogstad model is slightly more conservative than presented model; the difference between models vanishes as L increases as shown in Table 3 that gives the expectation of the maximum crest height $E(\eta_{c\ \max})$ for different values of the lifetime L. Note that $E(\eta_{c\ \max})$ is defined as the integral between 0 and ∞ of the probability of exceedance $P(\eta_{c\ \max} > \eta|[0, L])$.

Table 3. NOAA-NODC Buoy 46002: Comparison Between the Expectations of the Maximum Crest Height $E(\eta_{c\ \max})$, for Fixed Values of Lifetime L, Obtained With Krogstad and Presented Model

	Krogstad	Presented Model
10 years	14.7 m	14.0 m
100 years	16.9 m	16.5 m

[52] It is also noteworthy that, assuming that the occurrence of crest heights exceeding a fixed threshold is given by a Poisson process, we obtain a new expression of $P(\eta_{c\ \max} > \eta|[0, L])$ substituting the $R_J(\eta)$ expression (38) in equation (35).

[53] *Krogstad*'s [1985] expression (39) converges quickly to this new expression based on *Jasper*'s [1956] solution, as η increases. For this purpose we may consider, in equation (39), the following Taylor series expansion

$$\begin{aligned}\ln[1 - P(\eta_c > \eta)] = -P(\eta_c > \eta) - [P(\eta_c > \eta)]^2/2 + \\ - [P(\eta_c > \eta)]^3/3 - [P(\eta_c > \eta)]^4/4 \\ + O\left([P(\eta_c > \eta)]^5\right)\end{aligned} \quad (40)$$

[54] For example, for the buoy 46002, the maximum difference between Jasper and Krogstad probabilities $P(\eta_{c\ \max} > \eta|[0, L])$ are within 0.06% (0.03%) for L equal to 10years (100 years).

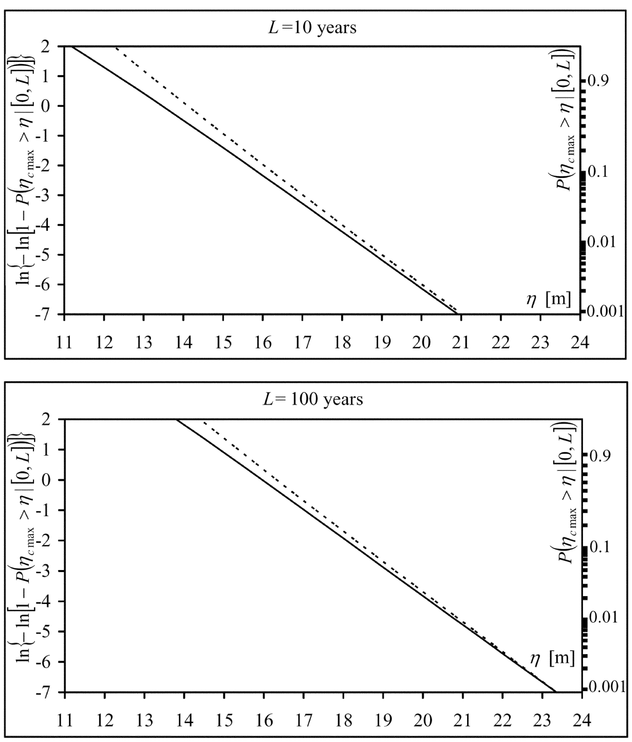

Figure 7. NOAA buoy 46002: the probability that maximum crest height, in the lifetime L, exceeds the fixed threshold η, calculated for lifetime equal to 10 and 100 years. Continuous lines are obtained with presented model (equation (27)); dotted lines are obtained with Krogstad model (equation (28)). The crest height distribution is given by equation (1).

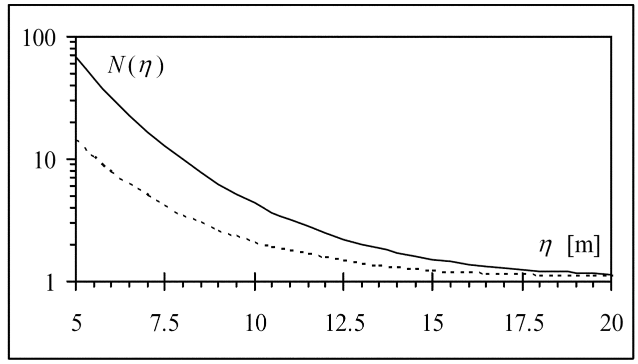

Figure 8. Mean number N of crest height exceeding the threshold η, in those storm in which the maximum crest height exceeds η: NOAA buoy 46002 (continuous line) and RON Ponza buoy (dotted line).

[55] Finally, to explain the difference with presented model, let us consider that, for a time span T', $T'/R_J(\eta)$ defines the mean number of crest heights exceeding the threshold η during T', while $T'/R(\eta)$ defines the mean number of sea storms in which the largest crest height exceeds the threshold η during T'. Therefore, $N = R(\eta)/R_J(\eta)$ defines the mean number of crest heights exceeding the threshold η, in those storms in which the maximum crest height exceeds η. Figure 8 shows values of N in two locations. In general, $T'/R_J(\eta) \geq T'/R(\eta)$, that is $N \geq 1$: in words the number of crests exceeding a fixed threshold η is greater than number of storms in which the maximum crest height exceeds η. For increasing η, $T'/R_J(\eta)$ approaches $T'/R(\eta)$ (that is N tends to 1). This explains also the difference between the probabilities $P(\eta_{c\ \max} > \eta|[0, L])$ given by Krogstad and by presented model: because $N \geq 1$ the Krogstad probability $P(\eta_{c\ \max} > \eta|[0, L])$ has to be greater or equal than results of presented model, for any η value. The difference between models decreases as η increases: they should give identical results in the limit as N tends to 1.

8. Conclusions

[56] A new solution for the return period of a sea storm in which the maximum nonlinear crest height exceeds a fixed threshold has been proposed. The general solution has been obtained by applying the Equivalent Triangular Storm model by *Boccotti* [1986, 2000]. Then, the nonlinear solution for the return period has been achieved by considering the *Forristall* [2000] second-order crest height distribution. The applications for three buoys moored off California and in Central Mediterranean Sea have shown that the height of extreme crest with the linear Rayleigh law is 8% smaller than with second order model in deep water, and that this difference slightly grows as the depth decreases.

[57] Finally, the comparison has been proposed between probabilities that maximum nonlinear crest height, in the lifetime L, exceeds a fixed threshold obtained with presented model and with *Krogstad* [1985] model. It is obtained that Krogstad approach is slightly conservative,

and that the difference between models tends to zero as lifetime increases.

[58] **Acknowledgment.** This work has been partially funded by the Italian National Research Group for Prevention from Hydro-Geological Disasters (GNDCI-CNR) (research team 1.45 of Reggio Calabria – grant for research 03.00013.GN42).

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