DATA SCIENCE

Two-Stage Stochastic Programming: Formulation of Markovitz's Model

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From Lesson 5 ...

- A further problem with the traditional Markowitz's model is the assumption on the return distribution
- In the model, only the first two moments of the distribution, i.e. the expected value and the variance, are considered while higher moments such as skewness and kurtosis are ignored.
- Traditional risk/return optimization therefore implicitly assumes that the returns follow a normal distribution.
- Empirical studies demonstrate, however, that normal distribution, especially during periods of market turbulence, significantly underestimates the likelihood of strongly negative returns.
- Empirical return distributions are not symmetric and exhibit fat tails.
- One way of resolving the problem is to incorporate forward-looking performance scenarios

Portfolio Optimization: Stochastic formulations

- This idea can be mathematically defined by means of the stochastic programming models
- We derive now a stochastic formulation of the Markowitz's model
- We shall assume that the random return $\widetilde{r}_i$ follows a discrete distribution
- We denote by S the number of realizations (scenarios)
- For each scenario s we denote by
  - r_{is} the s-th realization of the return of asset i
  - p_s the corresponding probability

Expected Value of the portfolio return

- We denote by $\widetilde{r}_p$ the random portfolio return with realization r_{ps}

$$r_{ps} = \sum_{i=1}^{N} r_{is} x_i$$

- The expected value of the portfolio return $\overline{r_p}$ is

$$E[\tilde{r}_p] = \overline{r}_p = \sum_{i=1}^{N} \overline{r}_i x_i$$

where

$$E[\widetilde{r}_i] = \overline{r}_i = \sum_{s=1}^{S} p_s r_{is}$$

Portfolio Variance

In order to define the variance of the portfolio return we introduce the scenario variables y_s

$$y^s = \sum_{i=1}^{N} r_{is} x_i - \sum_{i=1}^{N} \overline{r}_i x_i$$

The variance is then defined as

$$\sum_{s=1}^{S} p_s (y^s)^2$$

Stochastic Markowitz model

$$\min z = \sum_{s=1}^{S} p_s (y^s)^2$$

$$\sum_{i=1}^{N} \bar{r}_i x_i \geq \gamma$$

$$\sum_{i=1}^{N} x_i = 1$$

$$y^s = \sum_{i=1}^{N} r_{is} x_i - \sum_{i=1}^{N} \bar{r}_i x_i \quad s = 1, \dots, S$$

$$x_i \geq 0 \quad i = 1, \dots, N$$

$$y^s \quad \text{free} \quad s = 1, \dots, S$$

Remarks

- The variance penalizes the violations under and above the mean in the same way
- This measure is appropriate for symmetric distribution (as the Gaussian one), but not in general
- By using the stochastic programming approach other risk measures can be be used
  - Semi variance
  - Mad –Mean Absolute Deviation

Semivariance

- We consider only the violation under the mean
- The free variable y^s can be written as the difference of two nonnegative variables

$$y^s = y^{s+} - y^{s-}$$

where

$$y^{s+} = \sum_{i=1}^{N} r_{is} x_i - \sum_{i=1}^{N} \bar{r}_i x_i$$

$$y^{s-} = -\sum_{i=1}^{N} r_{is} x_i + \sum_{i=1}^{N} \bar{r}_i x_i$$

- The objective function is

$$\min z = \sum_{s=1}^{S} p_s (y^{s-})^2$$

- The mean absolute deviation is defined as

$$MAD(\tilde{r}_p) = E[|\underbrace{\tilde{r}_p - \bar{r}_p}_{y}|]$$

$$|y| := \begin{cases} y, & \text{if } y \geq 0, \\ -y & \text{if } y < 0 \ . \end{cases}$$

Mean Absolute Deviation

- In the case of discrete distribution

$$y^{s+} = \sum_{i=1}^{N} r_{is}x_i - \sum_{i=1}^{N} \overline{r}_i x_i$$

$$y^{s-} = -\sum_{i=1}^{N} r_{is}x_i + \sum_{i=1}^{N} \overline{r}_i x_i$$

- The objective function is

$$\min z = \sum_{s=1}^{S} p_s(y^{s-} + y^{s+})$$

Index Tracking Models

- All the measures introduced above consider the deviation from the expected value
- We may consider the deviation from a target - for example a market index
- In this case we may impose that

$$r_{ps} \geq I_s \qquad \forall s$$

- To repair for possible infeasibility we may consider the modified constraints

$$r_{ps} \geq T_s - \epsilon \qquad \forall s$$

and we may iteratively solve the problem by decreasing the ϵ value

An alternative formulation

- Until now, in our models we have considered as decision variables the fraction of capital invested in each asset
- An alternative formulation can be easily derived by assuming that the decision variables x_i represent the holding in a given asset i
- We denote by P_{i0} the price of asset i at time 0 and by P_{is} the future scenario prices
- By using this notation the basic formulation of the Markowitz model becomes

$$\sum_{i=1}^{N} P_{i0} x_i = B$$

where B denotes a given budget

An alternative formulation

- We may define the wealth under each scenario s

$$W^s = \sum_{i=1}^{N} P_{is} x_i$$

- The constraints on the expected return of the portfolio becomes

$$\bar{W} = \sum_{s=1}^{S} p_s W_s = \sum_{s=1}^{S} p_s (\sum_{i=1}^{N} P_{is} x_i) =$$

$$\sum_{i=1}^{N} \bar{P}_i x_i \geq \theta B$$

where $\bar{P}_i$ denotes the expected price computed by starting from the scenario prices and θ is a given threshold

An alternative formulation

- In order to compute the deviation, we consider the variables y^s

$$y^s = W^s - \bar{W}$$

- we may want to minimize the variance

$$\min \sum_{s=1}^{S} p_s (y^s)^2$$