n-Person Cooperative Games

and Geometrical Aspects:

A Brief Survey and New Perspectives

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Abstract

In this paper we consider cooperative games and various analytical tools in order to study behavioral situations of a set of players and to analyze different classes of games in a cooperative way. Following these approaches we can consider these classes of games as cones characterized by particular sets of generators.

1   Introduction and Preliminaries

Let n be a natural number greater than 1, $N = \{1, 2, \ldots, n\}$ and 2^N is the family of all subsets of N.

Definition 1.1 *An* n-person game *(in characteristic function form) is a pair (N, v), where v is a real-valued function on 2^N with $v(\emptyset) = 0$.*

The elements of N are called *players* and the elements of 2^N *coalitions* The function v is called *characteristic function* and for $S \in 2^N$, the real number $v(S)$ is called *worth* or *value* of the coalition S.

Definition 1.2 *An n-person game (N, v) is said to be in 0-1 normal form whenever $v(N) = 1$ and $v(\{i\}) = 0$ for $i = \{1, 2, \ldots, n\}$.*

An n-person game (N, v) is:

- *monotone* if and only if $T \subset S \subseteq N$ implies $v(T) \geq v(S)$;

- *simple* if and only if $v(S)$ is 0 or 1 for all $S \subseteq N$.

Definition 1.3 *Player i is a* veto *player for the simple n-person game (N, v) if and only if $v(S) - 1$ implies $i \in S$.*

We shall call a game (N, v):

- *additive* if $v(S \cup T) = v(S) + v(T)$ for all disjoint pairs $S, T \in 2^N$

- *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all disjoint pairs $S, T \in 2^N$

- *convex* if $v(S \cup T) + v(s \cap T) \geq v(S) + v(T)$ for all $S, T \in 2^N$

- *constant* if $v(S) + v(N - S) = v(N)$ for all $S, T \in 2^N$

- *simple* if $v(S) \in \{0, 1\}$ for all $S \in 2^N$

- *monotone* if $S \subset T$ implies $v(S) \leq v(T)$ for all $S, T \in 2^N$

- *symmetric* if $v(S) = v(T)$ for all $S, T \in 2^N$ with $|S| = |T|$

For a fixed n, an n-dimensional column vector $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbf{R}^n$ and $S \in 2^N \setminus \{\emptyset\}$ the sum $\sum_{i \in S} \mu_i$ will also denoted by $\mu(S)$ and $\mu(\emptyset) = \sum_{i \in \emptyset} \mu_i = 0$; μ is called an *imputation* if $\sum_{i=1}^{n} \mu_i = 1$ and $\mu_i > 0$ for $i = \{1, 2, \ldots, n\}$.

For an n-person game (N, v) the set

$$I(v) = \{\mu \in \mathbf{R}^n : \mu_i \geq v(\{i\}) \text{ for each } i \in N \text{ and } \mu(N) = v(N)\}$$

is called the *set of imputations.*

The subset $C(N, v)$ of $I(N, v)$ defined by

$$C(N, v) = \{\mu \in I(N, v) : \mu(S) \geq v(S), \text{ for each } S \in 2^N \setminus \{\emptyset\}\},$$

is the *core* of the game.

Lemma 1.4 *For every n-person game* (N, v) *we have* $C(N, v) \subseteq \mathbf{R}_+^n$.

Proof. Let $i \in N$ and $\mu \in C(N, v)$, then from the definition of core we have:

$$\mu = v(N) - \sum_{j \in N \setminus \{i\}} \mu_j \geq v(N) - v(N \setminus \{i\}) \geq 0.$$

So we have $\mu \in \mathbf{R}_+^n$.

The core is a compact convex polyhedron of dimension at most $n - 1$. It is contained in H^N (where H^N is the hyperplane in $\mathbf{R}^N$ defined by the equation $\mu(N) = v(N)$) and is bounded by the intersection of that hyperplane with other hyperplanes H^S, with $S \subset N$.

Example 1.5 *Let* $N = \{1, 2, \cdots, n\}$ *be a set of n-players. Using different social-economic characters of the players, we divide the set* N *in two disjoint parts,* N_0 *and* N_1*; we suppose also that the players are different firms and that the two parts of the set* N *are complementarity in the sense of the production of certain products. In such a way, if we consider a generic player (firm) in the subset* N_0*, for instance, the economical influence that he has in the market is not relevant. Conversely, if we consider a coalition of players in one of the two parts of* N *we can obtain more relevant economic situations. In order to obtain the last sentence we have to define a cooperative game* (N, v)*, where the characteristic function* v *is defined by:*

$$v(S) = \min\{|N_0 \cap S|, |N_1 \cap S|\}, \quad \text{for all } S \in 2^N.$$

The richness of papers in literature denotes the economic relevance of the notion of core in the economic context and its ductility in different applied problems. An important fact is, for instance, that if $\mu \in C(N, v)$ we can observe that does not exist any coalition $S \neq N$ which has the convenience to split itself since the payment $\mu(S)$ is greater than $v(S)$, the profit that all members of any subcoalition of S can be obtain. From these considerations, we can conclude that into the application of core in different economical problems it can be used to measure an infinite number of imputations, and it can be represented by a single vector μ or be the empty set.

In this context many authors had try to find different solutions using *Shapley value*[25], *Schmeidler core* or the "*tao vector of Tijs*" instead of core (for more details see for instance [23]).

Let $(S_1, S_2, \ldots . S_{2^n-n-2})$ be a sequence of proper subsets of N such that every element of the sequence which contain more than one player. In this case we can consider the game as a column vector in $\mathbf{R}^{2^n-n-2}$ of components $v(S_j)$, $j \in \{1, 2, \ldots, 2^n - n - 2\}$. So we define, for every player i, the n-person game (N, v_i) by:

$$v_i = \begin{cases} 1, & \text{if } i \in S \text{ and } |S| > 1 \\ 0, & \text{otherwise.} \end{cases}$$

The games obtained for each i are simple, monotone and every player in every game is a veto player.

In 1964 Aumann and Maschler prove the following result:

Theorem 1.6 *An n-person game (N, v) has nonempty core if and only if there exist nonnegative scalars $\beta_1, \beta_2, \ldots, \beta_n$ having $\beta_1 + \beta_2 + \ldots + \beta_n = 1$, and for which*

$$v \le \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n.$$

Geometrically this theorem means that the core $C(N, v)$ is nonempty if and only if v lies *on or below* the convex hull C_n of the games $v_1, v_2, \ldots, v_n$.

Remark 1.7 *We recall that the* convex hull *of a set $X \subseteq \mathbf{R}^n$, i.e. the intersection of all convex sets of $\mathbf{R}^n$.*

An unpublished result of L. S. Shapley is the following:

Theorem 1.8 *The set of all n-person games having nonempty core is the closed convex polyhedron $C_n - (\mathbf{R}^{2^n-n-2})_+$.*

2   Convex cones and hypercubes

In this section we recall some geometrical definitions and propositions helpful for the sequel.

Let V an n-dimensional Euclidian vector space with origin $\mathbf{O}$. Identify a generic vector $\mathbf{x}$ of V with the n-tuple $\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ of its coordinate with respect to a particular orthonormal base of V.

Definition 2.1 *A subset C of an n-dimensional vectorial space V is said a cone with vertex in $\boldsymbol{O}$, if $\boldsymbol{O} \in C$ and $\boldsymbol{x} \in C$ implies $\alpha\boldsymbol{x} \in C$, for every non-negative real scalar α.*
The particular cones consisting of a non-zero vector $\boldsymbol{x}$ and all its multiples $\alpha\boldsymbol{x}$ is called rays.

The set $C + \mathbf{x}^0$, where C is a cone and $\mathbf{x}^0$ is a vector of V is said *cone with vertex in the point x^0*; it is the set of all translations of a cone with vertex in O by the vector $\mathbf{x}^0$.

Definition 2.2 *A cone C is* convex *if the ray $\boldsymbol{x} + \boldsymbol{y}$ is in C whenever $\boldsymbol{x}$ and $\boldsymbol{y}$ are rays in C. Thus a set C of vectors is a* convex cone *if and only if it contains all vectors $\alpha\boldsymbol{x} + \beta\boldsymbol{y}$ for all $\alpha, \beta \geq 0$, $\boldsymbol{x}, \boldsymbol{y} \in C$.*

Of course we say convex cone every subspace of a Euclidian vector space which is both a cone and a convex set.

Theorem 2.3 *A subset C of a Euclidian vector space X is a convex come if and only if*

$$\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in C \ \ \text{for all} \ \ \boldsymbol{x}, \boldsymbol{y} \in C \ \ \text{and} \ \ \alpha, \beta \geq 0$$

Remark 2.4 *Cones generalized the notion of vector subspace and linear variety, which are convex cones.*

We denote by $C(X)$ the set of all elements obtained as a non-negative linear combination of points of the vector space $X \subset \mathbf{R}^n$. Then, we have that, if $X, Y \subset \mathbf{R}^n$, $C(X \cup Y) = C(X) + C(Y)$ and, if $X \subseteq C$, $C(X) \subseteq C$. If $C(X) = C$, X is said *generator set* of the cone C.

More relevant in our approach is the notion of *polyhedric cone*, defined as a cone obtained as a finite intersection of closed semispace. Very interesting is the Klee's definition of polyhedric cone:

Theorem 2.5 *[18] A cone C is polyhedral if and only it there exists a finite subset X di C such that every point of C is a non-negative linear combinational of points of X.*

In [29] the authors introduced, for a n-person game, the following notation:

- $b_i^V = v(N) - v(N - i)$, for each $i \in N$;

- $b^V = (b_1^V, \ldots, b_n^V)$; $R^V(S, i) = v(S) - b^V(S - i)$ for each $S \in 2^N$ and $i \in S$;

- $a_i^V = \max\{R^V(S, i) : S \in 2^N, i \in S\}$;

- $a^V = (a_1^V, \ldots a_n^V)$.

The number b_i^V is called the *marginal value of player i* in the game, the number $R^V(s, i)$ the *remainder in the coalition for the player i* and the number a^V is the *maximal remainder for the player i*. Using the previous notation we can define the *hypercube of the game* (N, v) as the set $H(v) = \{x \in \mathbf{R}^N : a_i^V \le x_i \le b_i^V \text{ for each } i \in N\}$.

We denote by G_N the $(2^N - 1)$-dimensional space of characteristic functions of n-person games and let H_N be the set of elements of G_N with non-empty hypercube.

A finite number of players in a determined social-economic space can create different scenarios of economic action following different approach: individually or coalitional, i.e. forming coalitions of agents - players. We denote by the symbol SG_N the set of game defined over the set of players N, where $N = \{1, 2, \ldots, n\}$. Let v_1 and v_2 be two characteristic functions defined in the same set N and λ be a scalar, then $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ and $(\lambda v)(S) = \lambda v(S)$ are characteristic functions too, defined over the set of parts of N, i.e. 2^N, with $S \subseteq N$. It follows that the set SG_N can be considered as a vectorial euclidian space type $\mathbf{R}^{2^N \setminus \{\emptyset\}}$ of dimension 2^{N-1}, where axis are indexed by non empty coalitions. In this research direction it is possible to analyze the previous notions both as for individual games than for unanimity games. In both situation the coordinates are *dividends* of coalitions (for more information see [23]).

We denote by SG_{saN} the set of superadditive games defined on N and contained in SG_N. If γ and ρ are two characteristic functions, α_1 and α_2 two non-negative scalars and $S, T \subseteq N$, with $S \cap T = \emptyset$ we have:

1. $\alpha_1 \gamma(\emptyset) + \alpha_2 \rho(\emptyset) = 0$

2. $(\alpha_1 \gamma + \alpha_2 \rho)(S \cup T) = \alpha_1 \gamma(S \cup T) + \alpha_2 \rho(S \cup T) \ge \alpha_1 \gamma(S) + \alpha_1 \gamma(T) + \alpha_2 \gamma(S) + \alpha_2 \rho(T) = (\alpha_1 \gamma + \alpha_2 \rho)(S) + (\alpha_1 \gamma + \alpha_2 \rho)(T)$.

The previous items 1 and 2 imply that the class SG_{saN} represent a (2^{N-1})-dimensional subcone of SG_N. It is easy to prove that the class of convex games belongs to the class of superadditive games.

3 Balanced-type games

More significant and relevant are classes of games for which hold a non-empty core. If we consider this class as a polyhedral cone, it must be emphasized the fact that the inequality that characterizes this type of cone is not treatable by an analytical point of view, especially in the case where n (the number of players) exceeds the number 4. For this reason in the last five decades, many authors have tried to overcome this difficulty trying to formalize a characterization of SGC_N, all of the above games with the core is not empty, seen as polyhedral cones. One of the first results obtained in this direction is due to Gurk [17] in 1959. A very important combinatorial characterization of games with non-empty core was given by Bondareva [3] and Shapley [26]. The core of a game appeared to be non-empty if and only the game is balanced. Furthermore, the set of characteristic functions of games with non-empty core appear to form a polyhedral cone. In 1971 Spinetto in [28] has given a geometric characterization by the following result:

Theorem 3.1 *The cone of non-negative games with non-empty core SGC_{N_+} is generated by simple games with players who have a veto power. Furthermore, these players and their non-negative multiples are the extreme elements of SGC_{N_+}.*

In 1986 Derks [6] proved that for any game $G \in \mathsf{SGC}_{N_+}$ there is a finite number of simple games with veto power ω_j and positive weights β_j, $j \in \{1, \ldots, n\}$, such that

$$G = \sum_{j=1}^{n} \beta_j \omega_j.$$

Let G be the family of subsets of the set $N = \{1, 2, \ldots, n\}$.

Definition 3.2 *[8] G is said to be* super-balanced *if there exists a system of positive weights $(\lambda_S)_{S \in G}$ such that*

$$\sum_{S \in G, i \in S} \lambda_S \geq 1, \quad \forall i \in N \tag{1}$$

Definition 3.3 *A family S is said to be* balanced *if there exists a system of positive weights $(\lambda_S)_{S \in G}$ such that $\forall i \in N$*

$$\sum_{S \in G, i \in N} \lambda_S = 1. \tag{2}$$

Remark 3.4 *A super-balanced family of coalitions is simply a covering of the grand coalition N.*

Definition 3.5 *A game is said to be* super-balanced *if for every super-balanced family $G \subset 2^N$, and for every associated system of weights $(\lambda_S)_{S \in G}$ one has*

$$\sum_{S \in G} \lambda_S v(S) \geq v(N). \tag{3}$$

Definition 3.6 *A game is* balanced game *if for every balanced family $G \subset 2^N$ and for every associated system of weights $(\lambda_S)_{S \in G}$ one has*

$$\sum_{S \in G} \lambda_s v(S) \leq v(N). \tag{4}$$

In 1982, Tijs and Lipperts introduced the following:

Definition 3.7 *[29] A game (N, v) is* semi-balanced *if the following system of inequalities holds:*

$$v(S) + \sum_{i \in S} v(N - i) \leq |S| v(N) \ \text{ for all} S \in 2^N - \{\emptyset\}.$$

Theorem 3.8 *[29] Every balanced game is semi-balanced.*

Definition 3.9 *A game is said* totally balanced *if every subgame of a game N is balanced.*

We denote these two classes of games by SGB_N and SGTB_N, respectively.

Theorem 3.10 *[29] $v \in H_n$ if and only if (N, v) is a semi-balanced game.*

Theorem 3.11 *[29] H_n is a polyhedral convex cone.*

Following this important results and considering our modeling we can observe that the class SGB_N is a cone. In fact let $C(N, v_1)$ and $C(N.v_2)$ be the cores of the games (N, v_1) and (N, v_2), respectively, $x \in C(N, v_1)$, $y \in C(N.v_2)$ and $\beta \in \mathbf{R}_+$, then we have $x + y \in C(N, v_1 + v_2)$ and $\beta x \in C(N, \beta v)$, Applying Bondareva and Shepley's result we can conclude that $\mathsf{SGB}_N = \mathsf{SGC}_N$.

To conclude this *excursus* it is important to cite a very important result concerning polyhedric cones and the cooperative games obtained by Derks in 1991:

Theorem 3.12 *[7] If (N, v) is a simple game where players have the veto power then this is an extremal element of the cone. Every element of SGB_{N_+} is a multiple of a simple game with veto power.*

Following Derk's methodology we can obtain a generic balanced and non-negative game as a non-negative linear combination of simple balanced games.

Open Problem. Is the set of characteristic functions of super-balanced game with non-empty core a polyhedral cone?

4   Conclusions

The aim of this paper is to show that the class of convex games can be viewed geometrically as a cone. In doing this we consider and collect some properties giving some remarks in order to emphasize the relationships that exist between some classes of *n*-person cooperative games and the polyhedral convex cones. By this approach, in the frame of balanced type games it could be interesting in analyzing the special case of super-balanced games by considering them in terms of a polyhedral cone. All these characterizations type could open new perspectives of research of the study concerning the role plays in this context of analysis by the characteristic functions (or value functions in some issues).

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